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SETTLING OF AN AXIALLY SYMMETRIC BODY IN A VISCOUS STRATIFIED FLUID

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Abstract—The uniform, upwards flow of a continuously vertically stratified fluid past an axially symmetric body is considered. The fluid is assumed to be Newtonian, incompressible, and diffusive. A matched asymptotic expansion procedure is used to calculate a correction to Stokes drag on the body. The results are valid provided that $\alpha \ll 1$, $\mathrm{Re} \ll \alpha^{1/3}$, $\mathrm{Fr}^2 \ll \alpha^{-1/3}$, $\mathrm{Pe} \gg \alpha^{2/3}$, where α is a stratification parameter. The results are applied to determine the quasi-steady motion of a body settling in a vertically stratified fluid.

1. INTRODUCTION

The steady, horizontal, slow viscous flows of vertically stratified fluids past obstacles have recently been investigated by several authors. Martin & Long (1968) solved for the horizontal flow past a flat plate. Graebel (1969) presented an analytical solution for a cylinder in a uniform horizontal translation of strongly stratified fluid and his results were confirmed by the experiments of Browand & Winant (1971). Janowitz (1971) treated the horizontal flow past a finite vertical two-dimensional plate. Chadwick & Zvirin (1974) evaluated the drag on a sphere in a horizontal flow of a vertically stratified fluid.

This present work deals with the settling of a small axially symmetric body in a vertically stratified diffusive fluid, where far from the body the density increases linearly with depth. As the body descends it encounters heavier fluid, making this problem time dependent. If the body is permitted to drop far enough it will come to rest at the level where its density equals that of the fluid. It is assumed that the Reynolds number is very small such that inertia terms can be neglected, and that the body settles very slowly, in a quasi-steady state. The drag on the body can thus be calculated at each instant of time as if the body were moving steadily at its instantaneous velocity.

We therefore consider the vertical (upwards) flow past a stationary body where the velocity far upstream is steady and uniform. A solution is sought which is hydrodynamically steady but allows for a time dependent buoyancy force.

For definiteness the drag force is initially calculated for a spherical solid, and the results are extended in Section 4 for other axially symmetric fluid or solid bodies.

The governing equations and the boundary conditions for the problem are, in a non-dimensional form and with a body-fixed coordinate system whose origin is at the center of the sphere:

$$-\nabla p + \nabla^2 \mathbf{q} - \alpha \rho \hat{\mathbf{i}} = 0 \qquad [1a]$$

$$\operatorname{div} \mathbf{q} = 0 \qquad [1b]$$

$$1 + \mathbf{q} \cdot \nabla \rho = \frac{1}{\mathrm{Pe}} \nabla^2 \rho \qquad [1c]$$

$$\mathbf{q} = 0 \quad \text{on} \quad r = 1 \qquad [2a]$$

$$\mathbf{q} \to \hat{\mathbf{i}} \quad \text{as} \quad r \to \infty \quad (r^2 = x^2 + y^2 + z^2) \qquad [2b]$$

$$\rho \to -x \quad \text{as} \quad r \to \infty. \qquad [2c]$$

The normalizations are defined as follows:

U—free stream velocity

a—radius of the sphere

$\rho'_o = \rho'(x = 0, y, z, t = 0)$ as $y^2 + z^2 \to \infty$—constant reference density

$r = r'/a$; $x = x'/a$—body-fixed vertical coordinate

$t = (U/a)t'$—the initial time $t' = 0$ is arbitrary

$\mathbf{q}(\mathbf{r}) = \mathbf{q}'(\mathbf{r})/U$

$\dfrac{\partial \rho'_\infty}{\partial X'} = -\rho'_o\beta$—constant density gradient far from the sphere; X' is the vertical space-fixed coordinate

$\rho(\mathbf{r}) + t = \dfrac{\rho'(\mathbf{r}, t) - \rho'_o}{\rho'_o a\beta}$

$p(\mathbf{r}) - \alpha t x = \dfrac{ap'(\mathbf{r}, t)}{\mu U} + \dfrac{\mathrm{Re}}{Fr^2}x$

$\mathrm{Re} = \rho'_o U a/\mu$—Reynolds number

$\mathrm{Fr} = U/(ga)^{1/2}$—Froude number

$\mathrm{Pe} = Ua/\mathscr{D}$—Peclet number

$\alpha = \dfrac{a\beta\,\mathrm{Re}}{\mathrm{Fr}^2} = \dfrac{\beta\rho'_o g a^3}{\mu U} > 0$—stratification parameter

$\hat{\mathbf{i}}$ = unit vector in the x direction.

The primed quantities ρ', p', and $\mathbf{q}$ are in physical units and have their usual hydrodynamic meanings; μ is the viscosity and $\mathscr{D}$ is the diffusivity.

An approximate solution of [1] and [2] is sought for $\alpha \ll 1$, which enables the problem to be treated by a perturbation technique, whereby the dependent variables are expanded in powers of α in inner and outer fields, similar to Chang (1960). The solution is valid for cases where $\mathrm{Pe} \gg \alpha^{2/3}$ which implies that diffusion does not dominate convection in the continuity equation. Finally, the conditions $\mathrm{Re} \ll \alpha^{1/3}$, and $\mathrm{Fr}^2 \ll \alpha^{-1/3}$ ensure that inertia terms can be neglected in both the inner and outer fields to all orders of α considered here (cf. Chadwick & Zvirin 1974). The drag force on the body is calculated by a method using Fourier transforms, as suggested by Childress (1964) and Chadwick & Zvirin (1974). Denoting the respective dimensional and non-dimensional Stokes drags by D'_s, D_s, where $D_s = D'_s/\mu U a$, the drag force on any axially symmetric body including the effects of stratification is given by:

$$\mathbf{F}' = D_s'\hat{\mathbf{i}}\left(1 + \frac{D_s}{6\pi}B\alpha^{1/3}\right) + 0(\alpha^{2/3}) \tag{3}$$

where B is a constant depending on $\gamma \equiv \alpha^{1/3}/\text{Pe}$. For the non-diffusive case ($\text{Pe} \to \infty$) $B = 1.060$, which is about seven times the value for the horizontal flow. Obviously the sphere creates a greater disturbance in the vertical flow than in the horizontal one. Because of the axial symmetry of the problem no moment will act on the sphere.

In Section 5 the equation of motion of the body is developed based on the drag formula [3]. The drag correction for the stratification effects leads to a nonlinear equation which is solved by a simple iterative procedure. The resulting solution for the settling velocity is valid provided the body is not too close to its final equilibrium position. Finally the theory is applied to the settling of small particles in a solar pond.

2. EXPANSION OF INNER AND OUTER FIELDS

A procedure of simple perturbation in α for solving [1]–[2] would fail because similar to Whitehead's paradox, the boundary condition at $r \to \infty$ cannot be satisfied by the particular integral of the first perturbation (cf. Van-Dyke (1964), p. 153). Therefore, a singular perturbation method is adopted, whereby the flow field is separated into inner and outer fields and their solutions are matched in an intermediate field. In the inner field the body force is small compared with the viscous forces, while in the intermediate field these forces have the same order of magnitude. From [1a] and [2c] it can be concluded that this happens when $r = 0(\alpha^{-1/3})$, which suggests the stretching factors given in [10]. The form of the inner expansions [5] and the outer expansions [11] is determined by the matching process in the intermediate field.

Because of the axial symmetry every property depends on r and x only. The expansions for the inner field are:

$$\mathbf{q}(r, x) = \mathbf{h}^{(o)}(r, x) + \alpha^{1/3}\mathbf{h}^{(1)}(r, x) + 0(\alpha^{2/3}) \tag{5a}$$

$$p(r, x) = p^{(o)}(r, x) + \alpha^{1/3}p^{(1)}(r, x) + 0(\alpha^{2/3}) \tag{5b}$$

$$\rho(r, x) = \rho^{(o)}(r, x) + \alpha^{1/3}\rho^{(1)}(r, x) + 0(\alpha^{2/3}). \tag{5c}$$

Introducing these expansions into [1] yields the following equations for the first two approximations:

$$-\nabla p^{(o)} + \nabla^2\mathbf{h}^{(o)} = 0 \tag{6a}$$

$$\operatorname{div}\mathbf{h}^{(o)} = 0 \tag{6b}$$

$$1 + \mathbf{h}^{(o)}\cdot\nabla\rho^{(o)} = \frac{1}{\text{Pe}}\nabla^2\rho^{(o)} \tag{6c}$$

$$\mathbf{h}^{(o)} = 0 \quad \text{on} \quad r = 1 \tag{7}$$

$$-\nabla p^{(1)} + \nabla^2\mathbf{h}^{(1)} = 0 \tag{8a}$$

$$\operatorname{div}\mathbf{h}^{(1)} = 0 \tag{8b}$$

$$\mathbf{h}^{(o)} \cdot \nabla \rho^{(1)} + \mathbf{h}^{(1)} \cdot \nabla \rho^{(o)} = \frac{1}{\text{Pe}} \nabla^2 \rho^{(1)} \qquad [8c]$$

$$\mathbf{h}^{(1)} = 0 \quad \text{on} \quad r = 1. \qquad [9]$$

It is noted that from [2] only the boundary conditions on the sphere [2a] are to be satisfied by the inner expansions, and the other boundary conditions will be fixed below by the matching.

The outer expansions are uniformly valid far from the sphere, i.e. as $r \to \infty$. In this region the sphere can be regarded as a point disturbance (or singularity) and outer co-ordinates are defined by:

$$\tilde{r} = \alpha^{1/3} r; \qquad \tilde{x} = \alpha^{1/3} x. \qquad [10]$$

The velocity, pressure and density are expanded by the following in the outer field:

$$\mathbf{q}(\tilde{r}, \tilde{x}) = \hat{\mathbf{i}} + \alpha^{1/3} \mathbf{g}^{(1)}(\tilde{r}, \tilde{x}) + 0(\alpha^{2/3}) \qquad [11a]$$

$$p(\tilde{r}, \tilde{x}) = \tfrac{1}{2} \alpha^{1/3} \tilde{x}^2 + \alpha^{2/3} p^{(1)}(\tilde{r}, \tilde{x}) + 0(\alpha) \qquad [11b]$$

$$\rho(\tilde{r}, \tilde{x}) = -\alpha^{-1/3} \tilde{x} + \rho^{(1)}(\tilde{r}, \tilde{x}) + 0(\alpha^{1/3}). \qquad [11c]$$

Thus the boundary conditions [2b] and [2c] are satisfied by the first terms in $\mathbf{q}$ and ρ.

Introduction of [10] and [11] into [1] yields the following set for the first perturbation of the outer field:

$$-\tilde{\nabla} p^{(1)} + \tilde{\nabla}^2 \mathbf{g}^{(1)} - \rho^{(1)} \hat{\mathbf{i}} = 0 \qquad [12a]$$

$$\operatorname{div} \mathbf{g}^{(1)} = 0 \qquad [12b]$$

$$\frac{\partial \rho^{(1)}}{\partial \tilde{x}} - \mathbf{g}^{(1)} \cdot \hat{\mathbf{i}} = \frac{\alpha^{1/3}}{\text{Pe}} \tilde{\nabla}^2 \rho^{(1)} \equiv \gamma \tilde{\nabla}^2 \rho^{(1)}. \qquad [12c]$$

[2b] and [2c] now lead to the following boundary conditions:

$$\rho^{(1)} = 0; \qquad \mathbf{g}^{(1)} = 0 \quad \text{as} \quad \tilde{r} \to \infty. \qquad [13]$$

In the following we consider the parameter γ to be of order of unity or less. The case of a non diffusive fluid is obtained as the special case $\gamma = 0$.

3. THE SOLUTIONS FOR THE FIRST TERMS IN THE EXPANSIONS

For the inner field, [6a] and [6b] yield the Stokes solution:

$$\mathbf{h}^{(o)} = \hat{i} - \frac{3}{2}\left(\frac{\hat{\mathbf{i}}}{r} - \nabla \frac{x}{2r}\right) + \frac{1}{4} \nabla \frac{\partial}{\partial x} \frac{1}{r} \qquad [14a]$$

$$p^{(o)} = -\frac{3x}{2r^3} \qquad [14b]$$

which satisfies the boundary condition [7] on the sphere and matches with the first terms of [11a] as $r \to \infty$. $\rho^{(o)}$ is then governed by the single equation [6c]. However, since $\mathbf{h}^{(1)}$

does not depend on $\rho^{(o)}$ it is not necessary to solve for $\rho^{(o)}$ in order to calculate the drag on the sphere to second approximation.

The intermediate variables are defined by:

$$r_\sigma = \alpha^{\sigma/3} r; \qquad x_\sigma = \alpha^{\sigma/3} x \tag{15}$$

where $0 < \sigma < 1$. Rewriting [14a] in these variables leads to:

$$\mathbf{h}^{(o)}(r_\sigma, x_\sigma; \alpha) = \hat{\mathbf{i}} - \frac{3}{2}\left(\frac{\hat{i}}{r_\sigma} - \nabla_\sigma \frac{x_\sigma}{2r_\sigma}\right)\alpha^{\sigma/3} + 0(\alpha^{2\sigma/3}). \tag{16}$$

For the outer field, as mentioned above, the sphere can be regarded as a singular point. This can be represented by the Stokes drag, $6\pi\delta(\mathbf{r})\hat{\mathbf{i}}$, on the right hand side of [12a]. Introducing the three dimensional Fourier transforms:

$$\begin{bmatrix} \mathbf{g}^{(1)} \\ p^{(1)} \\ \rho^{(1)} \end{bmatrix} = \frac{1}{8\pi^3}\int_{-\infty}^{\infty} e^{i\mathbf{k}\cdot\tilde{\mathbf{r}}} \begin{bmatrix} \mathbf{\Gamma}(\mathbf{k}) \\ \Pi(\mathbf{k}) \\ R(\mathbf{k}) \end{bmatrix} d\mathbf{k} \tag{17}$$

and substitution into [12a]–[12c] yields:

$$i\mathbf{k}\Pi + k^2\mathbf{\Gamma} + R\hat{\mathbf{i}} = -6\pi\hat{\mathbf{i}} \tag{18a}$$

$$\mathbf{k}\cdot\mathbf{\Gamma} = 0 \tag{18b}$$

$$ik_1 R - \mathbf{\Gamma}\cdot\hat{\mathbf{i}} = -\gamma k^2 R. \tag{18c}$$

The solutions for the last set are given by:

$$\mathbf{\Gamma}(\mathbf{k}) = \frac{6\pi}{k^2}\left\{-\hat{i} + \frac{(k^2 - k_1^2)\hat{\mathbf{i}} + (ik_1^2k^2 + \gamma k_1 k^4)\mathbf{k}}{k^2(1 + ik_1k^2 + \gamma k^4) - k_1^2}\right\} \tag{19a}$$

$$\Pi(\mathbf{k}) = -6\pi\frac{k_1^2k^2 - i\gamma k_1 k^4}{k^2(1 + ik_1k^2 + \gamma k^4) - k_1^2} \tag{19b}$$

$$R(\mathbf{k}) = -6\pi\frac{k^2 - k_1^2}{k^2(1 + ik_1k^2 + \gamma k^4) - k_1^2}. \tag{19c}$$

The formal solution for the outer field is given by insertion of [19] into [17]. In order to evaluate the drag on the sphere, however, it is sufficient to consider the expansion of the integral:

$$\frac{1}{8\pi^3}\int_{-\infty}^{\infty} (\mathbf{\Gamma} - \mathbf{\Gamma}_S)e^{i\mathbf{k}\cdot\tilde{\mathbf{r}}}\, d\mathbf{k} \quad \text{as} \quad \tilde{r} \to 0. \tag{20}$$

$\mathbf{\Gamma}_S$ is the Fourier transform of the fundamental Stokes solution:

$$\mathbf{\Gamma}_S(\mathbf{k}) = -\frac{6\pi}{k^2}\left(\hat{\mathbf{i}} - \frac{k_1\mathbf{k}}{k^2}\right). \tag{21}$$

The integral [20] is divided into two parts, $0 \le k \le \tilde{r}^{-\lambda}$, $0 < \lambda < 1$, where the exponent reduces to 1 and $k > \tilde{r}^{-\lambda}$, where k is large. Then, as $\tilde{r} \to 0$:

$$\frac{1}{8\pi^3}\int_{-\infty}^{\infty}(\mathbf{\Gamma}-\mathbf{\Gamma}_S)\,e^{i\mathbf{k}\cdot\tilde{\mathbf{r}}}\,d\mathbf{k} = \frac{1}{8\pi^3}\int_{k\leq\tilde{r}^{-\lambda}}(\mathbf{\Gamma}-\mathbf{\Gamma}_S)\,d\mathbf{k} + $$
$$+\frac{3}{4\pi^2}\int_{k>\tilde{r}^{-\lambda}}\frac{k^2(k^2-k_1^2)\hat{\mathbf{i}}-k_1(k^2-k_1^2)\mathbf{k}}{k^4(\gamma k^6+ik_1k^4)}e^{i\mathbf{k}\cdot\tilde{\mathbf{r}}}\,d\mathbf{k}+0(\tilde{r}^{1-\lambda}). \quad [22]$$

The first term on the right-hand side of [22] is a constant vector, denoted by **B**; the seond term is defined by **v**. Following Childress (1964) and Chadwick & Zvirin (1974), [22] can be written as:

$$\mathbf{g}^{(1)}-\mathbf{A}=\mathbf{B}+\mathbf{v} \quad \text{as} \quad \tilde{r}\to 0 \quad [23]$$

where

$$\mathbf{A}=-\frac{3}{2}\left(\frac{\hat{\mathbf{i}}}{\tilde{r}}-\tilde{\nabla}\frac{\tilde{x}}{2\tilde{r}}\right). \quad [24]$$

Rewriting [11a] and [23] in intermediate variables $r_\sigma=\alpha^{\sigma/3}r$, etc., we have for the outer expression:

$$\mathbf{q}(r_\sigma,x_\sigma;\alpha)=\hat{\mathbf{i}}-\frac{3}{2}\left(\frac{\hat{i}}{r_\sigma}-\nabla_\sigma\frac{x_\sigma}{2r_\sigma}\right)\alpha^{\sigma/3}+(\mathbf{B}+\mathbf{v})\alpha^{1/3}+o(\alpha^{1/3}). \quad [25]$$

Comparing [16] with [25] it is seen that the first two terms on the right-hand side of the latter are matched with $\mathbf{h}^{(o)}$. Thus $\mathbf{h}^{(1)}$ has to match $\mathbf{B}+\mathbf{v}$ in the intermediate field, or:

$$\mathbf{h}^{(1)}\to\mathbf{B}+\mathbf{v} \quad \text{as} \quad r\to\infty. \quad [26]$$

The first perturbation $\mathbf{h}^{(1)}$ to the inner velocity distribution is governed then by [8a], [8b], [9] and [26].

4. THE DRAG ON THE BODY

The drag force acting on the sphere is calculated by using Faxen's law, cf. Happel & Brenner (1965), p. 67, which states, in physical units:

$$\mathbf{F}'=6\pi a\mu(\mathbf{q}'_\infty)_{\text{or}}+\mu\pi a^3(\nabla'^2\mathbf{q}'_\infty)_{\text{or}} \quad [27]$$

where $\mathbf{q}'_\infty$ is the inner velocity far from the sphere and the subscript "or" denotes evaluation of the function at the origin. Since the inner solution far from the sphere is matched by the outer solution at the intermediate region which is far from the sphere, the drag is found by introduction of [25] and [26] into [27].

As can be seen from [22] **v** is expressed by an integral over large values of k. The two terms of [27] due to **v** are of the orders $0(\tilde{r}^{5\lambda})$ and $0(\tilde{r}^{3\lambda})$ respectively, $\lambda>0$. Thus the contribution of **v** to the drag, obtained by letting $\tilde{r}\to 0$ can be made arbitrarily small. The remaining term **B** is then determined from [19a] and [21]:

$$\mathbf{B}=\hat{\mathbf{i}}\frac{3}{4\pi^2}\int_{-\infty}^{\infty}\frac{(k^2-k_1^2)^3+\gamma k^6(k^2-k_1^2)^2}{k^4[(k^2-k_1^2+\gamma k^6)^2+k_1^2k^8]}\,d\mathbf{k}=\hat{\mathbf{i}}B \quad [28]$$

where the other components of the vector as well as its imaginary part vanish.

In order to evaluate B, the integral in [28] is transformed to spherical coordinates:

$$B(\gamma) = \frac{3}{\pi}\int_{\theta=0}^{\pi/2}\int_{k=0}^{\infty}\frac{\sin^5\theta(\sin^2\theta + \gamma k^4)}{(\sin^2\theta + \gamma k^4)^2 + k^6\cos^2\theta}\,\mathrm{d}k\,\mathrm{d}\theta. \qquad [29]$$

Numerical integration yields $B(\gamma)$ which is listed in table 1. It is noted that for the non-diffusive case ($\gamma = 0$) the integral is straight-forward and $B(0) = \Gamma(\frac{7}{3})\Gamma(\frac{1}{3})/2\Gamma(\frac{8}{3}) = 1.060$. As seen from the table, B decreases when γ increases; this indicates that the disturbance of the sphere becomes weaker when diffusion can take place.

For an axially symmetric body with dimensional Stokes drag $D_s'\hat{\mathbf{i}}$, the same analysis applies, but we replace the singularity $6\pi\delta(\tilde{\mathbf{r}})\hat{\mathbf{i}}$ in [12a] by $D_s\delta(\tilde{\mathbf{r}})\hat{\mathbf{i}}$ when solving the outer problem (cf. Chang 1960), which leads to the drag formula [3]. Payne & Pell (1960) have calculated D_s for a class of axially symmetric solid bodies. D_s for a fluid sphere is given in Happel & Brenner (1965). In general, even if $\mathbf{v}$ contributes to the drag, the generalized Faxen's law given by Hetsroni & Haber (1970) can be used.

5. SETTLING OF THE BODY

The equation of motion of the body when it settles in the linearly stratified fluid relates the acceleration to the forces acting on the body: gravity, buoyancy, and drag. If the body axis is aligned with the vertical, and the settling is slow enough such that a quasi-steady state prevails, then the drag is given by [3] at each instant during the descent. Denoting the depth of the body by $d(t)$, $d = d'/a$, the equation of motion in dimensionless form is (to order $0(\alpha^{1/3})$):

$$\ddot{d} + M_0[1 + \varepsilon \dot{d}^{-1/3}]\dot{d} + L_0 d = (1 - \eta)/\mathrm{Fr}_0^2 \qquad [30]$$

where

$$\left.\begin{aligned} &\eta = \frac{\rho_0'}{\rho_B'} < 1, \qquad M_0 = \kappa\eta/\mathrm{Re}_0, \qquad L_0 = \eta\alpha_0/\mathrm{Re}_0 \\ &\varepsilon = B(\gamma_0 \dot{d}^{-4/3})\alpha_0^{1/3} \ll 1 \\ &\kappa = D_{s,0}'/\mu U_0 \pi a s; \qquad \rho_B' = \text{density of body} \\ &s = \text{shape factor} \equiv V/\pi a^3 \\ &V = \text{volume of body}; \qquad a = \text{characteristic body dimension} \end{aligned}\right\}. \qquad [31]$$

The dot denotes differentiation with respect to the dimensionless time, and the subscript "o" implies that the quantity is evaluated at $t = 0$. The second term in the brackets of [30] arises from the dependence of α on velocity.

For the initial depth we take the reference point $d_0 = 0$, where the velocity of the body U_0 has already reached the terminal quasi-steady state:

Table 1. The dependence of the drag correction B on the diffusion parameter $\gamma = \alpha^{1/3}/\mathrm{Pe}$

γ	0.0	0.01	0.1	0.2	0.3	0.4	0.5	0.6	0.8	1.0	5.0	10.0
$B(\gamma)$	1.060	1.044	0.944	0.872	0.821	0.782	0.750	0.723	0.681	0.649	0.442	0.372

$$d = 0; \qquad \dot{d} = 1 \quad \text{at} \quad t = 0. \tag{32}$$

The equation of motion [30], is nonlinear due to the effects of stratification. However, the presence of the small parameter ε suggests an expansion of the type

$$d = d^{(o)} + \varepsilon d^{(1)} + \cdots \tag{33}$$

which permits the linearization of [30].

Substitution of [33] into [30] gives the following equations for $d^{(o)}$ and $d^{(1)}$:

$$\ddot{d}^{(o)} + M_o\dot{d}^{(o)} + L_o d^{(o)} = (1 - \eta)\frac{1}{\mathrm{Fr}^2} \tag{34a}$$

$$d^{(o)} = 0; \qquad \dot{d}^{(o)} = 1 \quad \text{at} \quad t = 0 \tag{34b}$$

$$\ddot{d}^{(1)} + M_o\dot{d}^{(1)} + L_o d^{(1)} = -M_o(\dot{d}^{(o)})^{2/3} \tag{35a}$$

$$d^{(1)} = 0; \qquad \dot{d}^{(1)} = 0 \quad \text{at} \quad t = 0. \tag{35b}$$

It is immediately seen that the particular solution of [34a]:

$$d_{\mathrm{eq}}^{(o)} = \frac{1-\eta}{\eta}\frac{\mathrm{Re}_o}{\alpha_o \mathrm{Fr}_o^2} \quad \text{or} \quad d_{\mathrm{eq}}^{(o)\prime} = \frac{\rho_B' - \rho_o'}{\beta\rho_o'} \tag{36}$$

is the asymptotic equilibrium depth where the densities of the body and fluid are equal.

The complete solution of [34a, b] is

$$d^{(o)} = d_{\mathrm{eq}}^{(o)} + c_1 e^{\phi_1 t} + c_2 e^{\phi_2 t} \tag{37}$$

where

$$\phi_{1,2} = -\tfrac{1}{2}M_o[1 \mp (1 - 4L_o/M_o^2)^{1/2}]$$
$$\simeq -L_o/M_o, \qquad -M_o \tag{38a}$$

$$c_1 = \frac{1 + \phi_2 d_{\mathrm{eq}}^{(o)}}{\phi_1 - \phi_2} \tag{38b}$$

$$c_2 = \frac{1 + \phi_1 d_{\mathrm{eq}}^{(o)}}{\phi_1 - \phi_2}. \tag{38c}$$

From [31] it is clearly seen that under the assumptions made in Section 1, $4L_o/M_o^2 \ll 1$. This implies that $\phi_{1,2}$ are real and negative and the body approaches the equilibrium state [36] without oscillations. Moreover, the second exponential in [37], $e^{\phi_2 t}$, decays much faster than the first one. Thus the zeroth order quasi-steady settling can be approximately described by

$$d^{(o)} \simeq d_{\mathrm{eq}}^{(o)}(1 - e^{-t/\tau}); \qquad \tau = M_o/L_o \tag{39}$$

which leads to the zeroth order settling velocity:

$$\dot{d}^{(o)} \simeq \frac{d_{\mathrm{eq}}^{(o)}}{\tau}e^{-t/\tau} = \frac{U_{s,o}}{U_o}e^{-t/\tau} \tag{40}$$

where

$$U_{s,o} = \frac{a^2 g(\rho'_B - \rho'_o)}{\mu\kappa}$$

which is the terminal Stokes settling velocity evaluated for the conditions at $t = 0$. Equation [40] is accurate for $t \gg M_o^{-1}$ and $d_{eq}^{(o)} \gg 1$.

Substitution of [40] into [35a] and using [35b] gives the first order correction to the settling velocity

$$\dot{d}^{(1)} = \left(\frac{U_{s,o}}{U_o}\right)^{2/3} (2e^{-2t/3\tau} - 3e^{-t/\tau}). \qquad [41]$$

Finally [41], [40], and [33] lead to

$$\dot{d} = \frac{U_{s,o}}{U_o} e^{-t/\tau} \left[1 + \varepsilon(2e^{t/3\tau} - 3)\left(\frac{U_{s,o}}{U_o}\right)^{-1/3}\right] + 0(\varepsilon^2). \qquad [42]$$

There are several points which must be discussed concerning the utility of [42]. Firstly, the initial velocity of the body, U_o, is not in general a known quantity, and it must be determined by experiment. However, for a large class of practical problems, when $M_o \gg 1$, $U_o \simeq U_{s,o}$. Secondly, the small parameter ε as defined by [31] is a function of time because of the functional dependence of B on the velocity. If diffusion is present ($\gamma_o \neq 0$), it is sufficiently accurate to use $\gamma_o[\dot{d}^{(o)}]^{-4/3}$ as the argument of B. Thirdly, [42] is valid for t in the range

$$M_o^{-1} \ll t \ll 3\tau \ln\left(\frac{1}{2\varepsilon}\right).$$

The lower limit as already mentioned, allows the second exponential in [37] to be neglected; the upper limit ensures that the second term in the expansion [33] is smaller than the first, and that $\alpha \ll 1$. Since $\varepsilon \ll 1$, the upper limit imposes a restriction on the use of [42] only in the immediate neighborhood of $d_{eq}^{(o)}$.

The theory presented here necessarily results in small changes, by the inherent nature of the perturbation process; nevertheless it is of interest to apply the theory to a specific problem. As an example of an application, consider the settling of particles in a solar pond, which is stably stratified with dissolved salts to prevent convection. The ability of the lower surface of the pond to absorb energy is strongly affected by the presence of small pollutant particles. It is therefore important to know how long it takes the particles to reach the bottom of the pond.

We consider a spherical particle with a radius of 10^{-5} m, and a density of 1.5×10^3 kg/m^3. The pond is 1 m deep and is linearly stratified with a density of 1.0×10^3 kg/m^3 at the top and increasing to 1.3×10^3 kg/m^3 at the bottom. Taking a viscosity $\mu \sim 10^{-3}$ kg/m sec, the terminal Stokes settling velocity $U_{s,o}$ is 1×10^{-4} m/sec. We also calculate $\mathrm{Re}_o = 1 \times 10^{-3}$, $\mathrm{Fr}_o^2 = 1 \times 10^{-4}$, $a\beta = 3 \times 10^{-6}$, $\alpha_o = 3 \times 10^{-5}$, $\alpha_o^{1/3} = 3.1 \times 10^{-2}$, so that the parameter restrictions stated in Section 1 are satisfied. Based on $U_{s,o}$ the particle reaches the bottom of the pond in 1.0×10^4 sec. The zeroth order correction [39] increases the settling time to 1.37×10^4 sec. Finally, the first order correction [42] increases the time to slightly over 1.42×10^4 sec.

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Résumé—On considere l'écoulement vers le haut d'un fluide continument stratifié verticalement sur un corps axisymmetrique. Le fluide est supposé être newtonien, incompressible et soumis au phénomène de diffusion. Un'procêdé équivalent à une expansion assymptotique est utilisé pour calculer la correction du drainage de Stokes sur le corps. Les résultats ne sont valables que pour $\alpha \ll 1$, $Re \ll \alpha^{1/3}$, $Fr^2 \ll \alpha^{-1/3}$, $Pe \gg \alpha^{2/3}$, ou α represente le coefficient de stratification. Les résultats sont appliqués pour determiner le mouvement presque continu d'un corps exposé à un fluide verticalement stratifié.

Auszug—Die gleichfoermige Aufwaertsstroemung einer kontinuierlich vertikal geschichteten Fluessigkeit um einen rotationssymmetrischen Koerper wird betrachtet. Die Fluessigkeit wird als Newtonisch, unzusammendrueckbar, und diffusiv angenommen. Mit einer Methode angepasster asymptotischer Expansion wird ein Korrekturglied zum Stokesschen Widerstand am Koerper berechnet. Die Gueltigkeit der Ergebnisse setzt $\alpha \ll 1$, $Re \ll \alpha^{1/3}$, $Fr^2 \ll \alpha^{-1/3}$, und $Pe \gg \alpha^{2/3}$ voraus, wobei α ein Schichtungsparameter ist. Die Resultate werden zur Bestimmung der quasistationaeren Bewegung eines in einer vertikal geschichteten Fluessigkeit absinkenden Koerpers benuetzt.

Резюме—Рассмотрено однородное восходящее течение непрерывной жидкости, расслоеннной в вертикальном направлении, прошедшей осесимметричное тело. Предположено, что жидкость нъютоновская, несжимаемая и диффчзная. Процдчра соответствующего ассимптотического раскрытия использует расчет поправки на закон Стокса для указанного тела. Результаты действительны при условии, что параметр расслоения α значительно менее единицы, критерий Рейнольдса $Re \ll \alpha^{1/3}$, критерий фруда $Fr^2 \ll \alpha^{-1/3}$, критерий Пекле $Pe \gg \alpha^{2/3}$. Достигнутые результаты применены к псевдоустановившемуся движению оседающего тела в вертикально расслоенной жидкости.